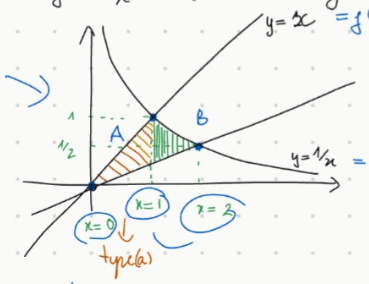
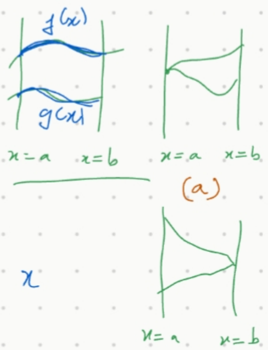
Find the area enclosed by

$y = \frac{1}{x}$, $y = x$, $y = \frac{1}{4}x$

A) Integrating with x





1) Solve for intersection points of curves

$f(x) = g(x)$

$f(x) = h(x)$

$g(x) = h(x)$

x

a) $\quad x = \frac{1}{4}x \Rightarrow x = 0$

b) $\quad x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = 1 \quad (x > 0)$

c) $\quad \frac{1}{x} = \frac{1}{4}x \Rightarrow x^2 = 4 \Rightarrow x = 2 \quad (x > 0)$

2) Split the region into smaller regions such that on each region you have unique $f(x), g(x)$.

Thus the area = area of A + area of B

$$= \int_0^1 |f(x) - g(x)|\,dx + \int_1^2 |h(x) - g(x)|\,dx$$

3) Compute the integral

$$\text{Area} = \int_0^1 \left|x - \frac{1}{4}x\right| dx + \int_1^2 \left|\frac{1}{x} - \frac{1}{4}x\right| dx$$

$$= \int_0^1 \frac{3}{4}x\,dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$$

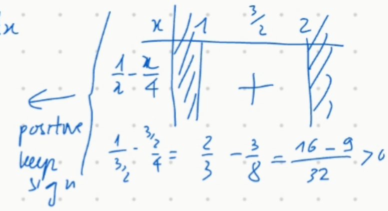
$$= \frac{3}{4}\frac{x^2}{2}\Big|_0^1 + \left(\ln|x| - \frac{1}{4}\frac{x^2}{2}\right)\Big|_1^2$$

$$= \frac{3}{4}\frac{1}{2} + \left[\left(\ln 2 - \frac{1}{4}\frac{\cancel{2^2}}{2}\right) - \left(\ln 1 - \frac{1}{8}\right)\right]$$

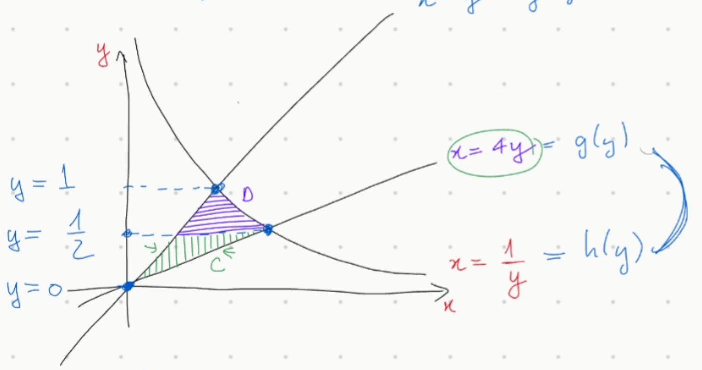
$$= \frac{3}{8} + \ln 2 - \frac{1}{2} - 0 + \frac{1}{8}$$

$$= \frac{4}{8} - \frac{1}{2} + \ln 2$$

$$= \ln 2$$



B (Integrating with y)



1) Solve for intersection points in variable y
write functions in the form
$x = f(y)$

$\Rightarrow$ $x = y = f(y)$
$x = 4y = g(y)$
$x = \frac{1}{y} = h(y)$

a) $f(y) = g(y)$
$\Rightarrow y = 4y \Rightarrow y = 0$

b) $f(y) = h(y)$
$\Rightarrow y = \frac{1}{y} \Rightarrow y^2 = 1 \Rightarrow y = 1 \quad (y > 0)$

c) $g(y) = h(y)$
$\Rightarrow 4y = \frac{1}{y} \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \frac{1}{2} \quad (y > 0)$

2) split the region into 2 pieces. C and D

C is enclosed by curves $\begin{cases} x = 4y \\ x = y \end{cases}$ $\begin{cases} y = 0 \\ y = 1/2 \end{cases}$

$$\text{Area of } C = \int_0^{1/2} |4y - y|\, dy = \int_0^{1/2} 3y\, dy = 3\frac{y^2}{2}\Big|_0^{1/2} = \frac{3}{8}.$$

D is enclosed by curves $\begin{cases} x = \frac{1}{y} \\ x = y \end{cases}$ $\begin{cases} y = \frac{1}{2} \\ y = 1 \end{cases}$

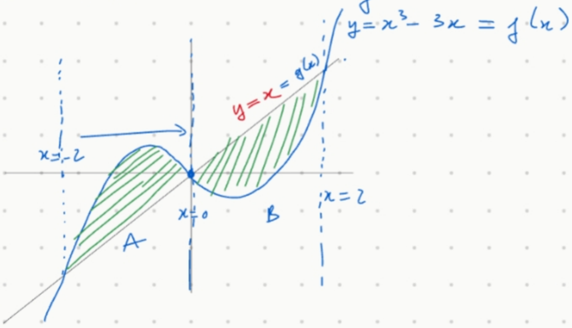
$$\begin{aligned} \text{Area of } D &= \int_{1/2}^{1} \left|\frac{1}{y} - y\right| dy \\ &= \int_{1/2}^{1} \left(\frac{1}{y} - y\right) dy \\ &= \ln|y| - \frac{y^2}{2}\Big|_{1/2}^{1} \\ &= \left(\ln 1 - \frac{1}{2}\right) - \left(\ln\frac{1}{2} - \frac{(1/2)^2}{2}\right) \\ &= -\frac{1}{2} - \ln 2^{-1} + \frac{1}{8} \\ &= -\frac{1}{2} + \frac{1}{8} + \ln 2 \end{aligned}$$

$\frac{1}{y} - y$ | $\frac{3}{4}$ | 1
$+$

$\frac{1}{\frac{3}{4}} - \frac{3}{4} = \frac{4}{3} - \frac{3}{4} = \frac{16-9}{12} > 0$

$$\begin{aligned} \text{Area of the region} &= \text{area of } C + \text{area of } D \\ &= \frac{3}{8} + \left(-\frac{1}{2} + \frac{1}{8} + \ln 2\right) \\ &= \ln 2 \end{aligned}$$

Example: Find the area of the shaded region



1) Find intersection points.

$$x = x^3 - 3x$$
$$\Rightarrow \quad x^3 - 4x = 0$$
$$\Rightarrow \quad x(x^2 - 4) = 0$$
$$\Rightarrow \quad \left[\begin{array}{l} x = 0 \\ x^2 = 4 \end{array}\right. \Rightarrow \left[\begin{array}{l} x = 0 \\ x = \pm 2 \end{array}\right.$$

2) Area of the region = area of A + area of B

A is enclosed by $\begin{cases} y = x^3 - 3x \\ y = x \end{cases}$ $\begin{cases} x = -2 \\ x = 0 \end{cases}$

$$\text{Area of } A = \int_{-2}^{0} |x^3 - 3x - x|\, dx$$
$$= \int_{-2}^{0} |x^3 - 4x|\, dx$$
$$= \int_{-2}^{0} (x^3 - 4x)\, dx$$
$$= \left(\frac{x^4}{4} - \frac{4x^2}{2}\right)\Big|_{-2}^{0}$$
$$= 0 - \left(\frac{2^4}{4} - 4\frac{2^2}{2}\right)$$
$$= -(4 - 8)$$
$$= 4$$

x	-2	-1	0
$x^3 - 4x$		positive	

$-1 + 4 = 3 > 0$

B is enclosed by $\begin{cases} y = x^3 - 3x \\ y = x \end{cases}$ $\begin{cases} x = 0 \\ x = 2 \end{cases}$

$$\text{area of } B = \int_0^2 |x^3 - 3x - x|\, dx = \int_0^2 |x^3 - 4x|\, dx$$
$$= -\int_0^2 (x^3 - 4x)\, dx$$
$$= -\left(\frac{x^4}{4} - 2x^2\right)\Big|_0^2$$
$$= -\left(\frac{2^4}{4} - 2 \cdot 2^2\right)$$
$$= -(4 - 8)$$
$$= 4$$

x	0	1	2
$x^3 - 4x$		$-$	

$1^3 - 4 = -3$

$$\text{Area} = \text{Area of } A + \text{area of } B = 4 + 4 = 8$$

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§ 6.2. Volumes

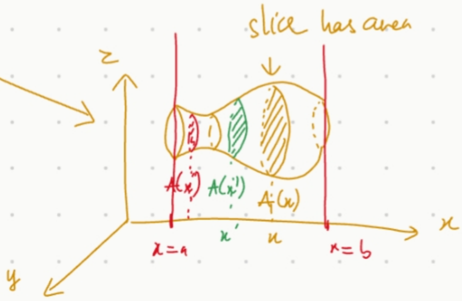
Volume,

Start with cylinder solids:

Volume = area of the base × height.

" Volume = sum of sections "

= " integrals of slices "

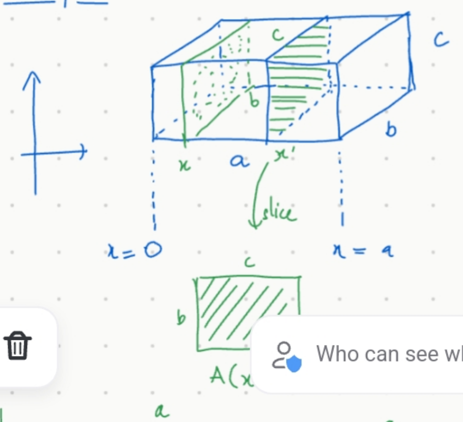


different slices has different areas.

Theorem Volume of solid lies between $x=a$, $x=b$ with cross-sectional area $A(x)$, A is a continuous function

$$V = \int_a^b A(x)\, dx.$$

Example 1. Box



Volume of the box

$= a \cdot b \cdot c.$

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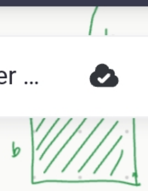


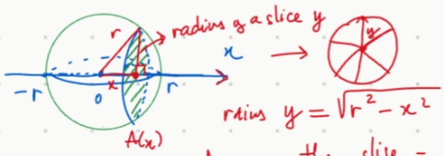
$A(x) = b \cdot c$

Theorem

$\Rightarrow \quad V = \int_0^a A(x)\,dx = \int_0^a bc\,dx = bc\int_0^a dx = bcx\Big|_0^a = abc.$

Example 2. volume of sphere of radius r.

$x = -r,\ x = r.$



radius $y = \sqrt{r^2 - x^2}$

Area of the slice $= \pi y^2 = \pi(r^2 - x^2) = A(x)$

Volume of sphere $= \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$

$= \pi\left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r}$

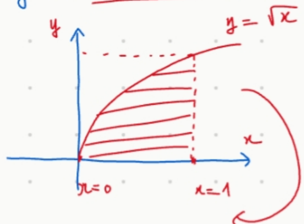
$= \pi\left(r^3 - \frac{r^3}{3}\right) - \pi\left(r^2(-r) - \frac{(-r)^3}{3}\right)$

$= \pi\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right)$

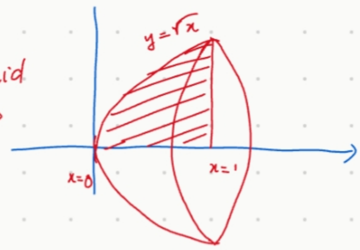
$= \frac{4}{3}\pi r^3$

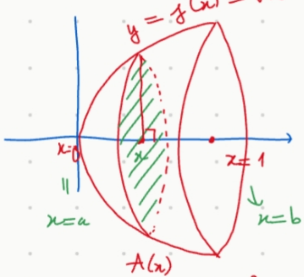
Example: Volumes of solids of Revolution.

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$.



solid →





the slice at x is a disk center at x of radius $f(x)$

radius ↓

the slice has area $= \pi(f(x))^2$

$\Rightarrow \quad A(x) = \pi\, f(x)^2$

... the solid of ...

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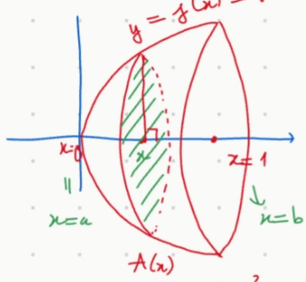

the slice at x is a disk
center at x of radius $f(x)$

radius

the slice has area $= \pi (f(x))^2$

$\Rightarrow A(x) = \pi \cdot f(x)^2$

Volume of the solid of revolution

$$= \int_a^b A(x)\,dx = \int_a^b \pi f(x)^2\,dx$$

$$= \int_0^1 \pi (\sqrt{x})^2\,dx$$

$$= \pi \int_0^1 x\,dx$$

$$= \pi \left.\frac{x^2}{2}\right|_0^1$$

$$= \frac{\pi}{2}$$

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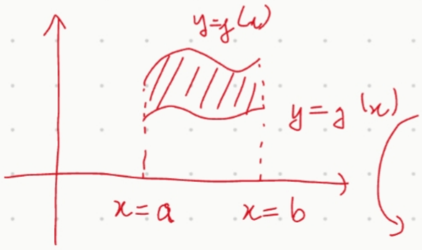
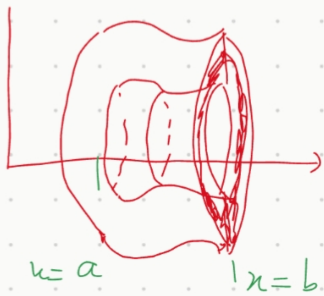
General volume of solids of revolution.

$$\begin{cases} y = f(x) \\ y = g(x) \end{cases} \qquad \begin{cases} x = a \\ x = b \end{cases}$$



rotate

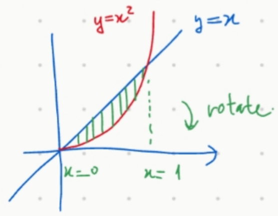
solid →



Volume of the solid

$$= \pi \int_a^b \left| f(x)^2 - g(x)^2 \right| dx$$

Example: The region R enclosed by $y = x$, $y = x^2$ is rotated about the x axis. Find the volume.



Volume of the solid $= \pi \int_0^1 |f(x)^2 - g(x)^2| \, dx$

$$= \pi \int_0^1 |x^2 - x^4| \, dx$$

$$= \pi \int_0^1 (x^2 - x^4) \, dx$$

$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$= \frac{2}{15} \pi .$$